Visual e-Assessment with JSXGraph in Calculus

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JSXGraph and STACK used in MA

IDIAM and multivariate analysis as a trigger for 3D development in JSXGraph. Some topics in engineering mathematics require the ability to visualise objects in 2D or 3D space.

- Focus on integration domains and extrema of functions in 2D
  - Coordinate transformations (polar coordinates, spherical coordinates)
  - Calculus of functions with two variables (negative definite Hessian)
  - Visualization of vector fields in 3D (curl of a vector field)

Ongoing ideas

- Work inspired by IDIAM
    - Application to implicit curves: constraint optimisation or Lagrangian multiplier role
    - Slope fields and trajectories

Examples available at the IDIAM Page

JSXGraph 3D development and most examples have been funded by ERASMUS+ “Interactive Digital Assessments in Mathematics”.

Integration in 2D/3D

Given $G \subset \mathbb{R}^n$ $(n = 2, 3)$. The integral wrt. G and a function $f : \mathbb{R}^n \to \mathbb{R}$ integrable on G is denoted as

$$\int_G f(\mathbf{x})\, \mathrm{d}G.$$

How to compute this?

G may given by two functions

$$G = \{(x,y) \in \mathbb{R}^2 \,|\, a < x < b,\ y_2(x) < y < y_1(x)\}$$

then

$$\int_G f(\mathbf{x})\,\mathrm{d}G = \int_a^b \int_{y_2(x)}^{y_1(x)} f(x,y)\,\mathrm{d}y\mathrm{d}x.$$

> Recover the functions y_1, y_2 from a diagram is demanding for some students.

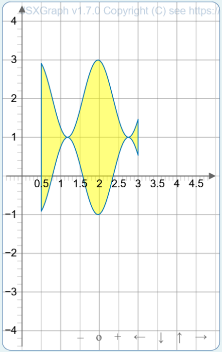
Given is a region of type

Question is missing tests or variants.

$$G = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ y_2(x) \leqslant y \leqslant y_1(x)\}$$

as shown in the diagram. Determine the interval $[a, b]$ and the expressions for the graphs of functions y_1 and y_2.

Give all numerical values as fractions instead of decimal numbers e.g. `1/2` instead of `0.5`.



$[a, b]$ =

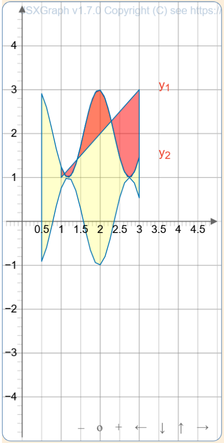
$y_1(x)$ =

$y_2(x)$ =

Check

Students view

This diagram shows the domain you have entered and the domain asked for. The given area is colored yellow, the red area results from your answer. The functions y_1 and y_2 have been labeled according to your input.



The value you gave for x_1 is not correct.

Nice, you found the correct value for x_2! Good job!

Check whether you did anything different here than for x_1 and try again.

Feedback view

Integration

The **Transformation Theorem** is widely used in integration.

> Given two sets $G \subset \mathbb{R}^n$ and $H \subset \mathbb{R}^n$ im $\mathbb{R}^n$ and a one-to-one mapping $T : H \to G$

$$T(\mathbf{u}) := \mathbf{x}(\mathbf{u}).$$

> T is continuously differentiable and $\det(J_T(u, v, w)) \neq 0$ on H. **Then**

$$\int_G f(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int_H f(\mathbf{x}(\mathbf{u}))\left|\det J_T(\mathbf{u})\right|\ \mathrm{d}\mathbf{u}.$$

Integration in 2D (Polar coordinate)

The introductory example in 2D integration are the Polar Coordinates:

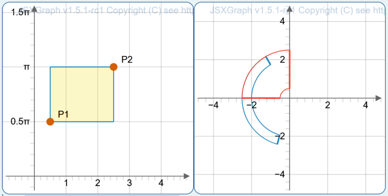
$$T : [0,\infty) \times [0,2\pi) \to \mathbb{R}^2 \text{ with } \begin{pmatrix} r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}$$

Tool zum Nachbessern der Frage | ! Es fehlen Tests oder Varianten.

Given is a 2D area with polar geometry. It is defined by the intervals for each of the polar coordinates r and ϕ. Here r is the radial coordinate and ϕ is the angle starting at the x-axis oriented counterclockwise with $\phi \in [0, 2\pi]$.

Reconstruct the intervals that define the given area by matching the areas using the cartesian coordinate system.

Write the interval in the form $r \in$`[r1,r2]` and $\phi \in$`[phi1,phi2]`, e.g. `[1/2,2]` and `[1/2*pi,2*pi]`.



$r \in$ [ ]

$\phi \in$ [ ]

Students view

Integration in 3D

Spherical coordinates are very challanging for the students. The question: Describe the set M given by

$$M := \{\mathbf{x} \in \mathbb{R}^3 \,:\, \|\mathbf{x}\|_2 \leqslant 1,\, x_1, x_2, x_3 < 0\}$$

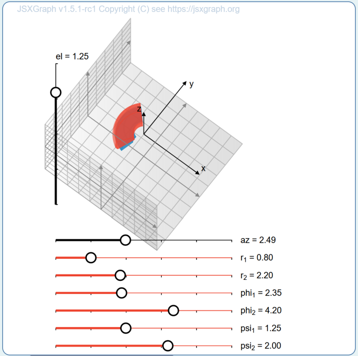
in spherical coordinates could only be solved by a few students.

Spherical Coordinates

Given is a 3D volume with spherical geometry. It is defined by the intervals for each of the spherical coordinates r, ϕ and ψ. Here r is the radial coordinate and ϕ is the azimuthal angle starting at the x-axis oriented counterclockwise with $\phi \in [0, 2\pi]$. Lastly, ψ is the polar angle measured from the z-axis with $\psi \in [0, \pi]$.

Tool zum Nachbessern der Frage | Es fehlen Tests oder Varianten.

Reconstruct the intervals that define the given volume.



$r_1 =$ 0.8

$r_2 =$ 2.2

$\phi_1 =$ 2.35

$\phi_2 =$ 4.2

$\psi_1 =$ 1.25

$\psi_2 =$

Students view

Spherical Coordinates

Your answer is partially correct.

The value you gave for r_1 is not correct.

Nice, you found the correct value for r_2! Good job!

Check whether you did anything different here than for r_1 and try again.

Correct answer, well done.

Nice, you found the correct value for ϕ_1! Good job!

Nice, you found the correct value for ϕ_2! Good job!

Perfect! You got both values of ϕ right!

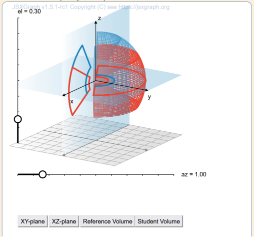
Your answer is partially correct.

The value you gave for ψ_1 is not correct.

Nice, you found the correct value for ψ_2! Good job!

Check whether you did anything different here than for ψ_1 and try again.

You can now compare your response to the reference volume. Your solution is displayed in orange. In addition, you can see the cross sections in the $x-y$-plane and $x-z$-plane. Note, that you can deactivate the visualizations using the button.



Feedback view

Curl

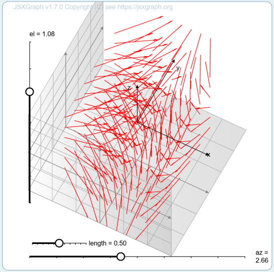
The curl of a vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ is just $\operatorname{curl} V = \nabla \times V$.
It is quite strong to get an idea of it.

Question is missing tests or variants.

Given is the curl of a vector field $\hat{V} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$\hat{V}(x, y, z) := \begin{pmatrix} 1 - x \\ z \\ z \end{pmatrix}$ as shown in the diagram.

Select the vector field V, so that $\hat{V} = \nabla \times V$ is valid.



Select V.

- ○ (Clear my choice)

- ○ $[z, x, y]$
- ◉ $[y, -x, 0]$
- ○ $\left[\frac{z^2}{2}, x \cdot z, y\right]$

$[y, -x, 0]$

Check

Curl

❌ Incorrect answer.

The entries underlined in red below are those that are incorrect.

$$\left[\underline{y}, \underline{-x}, \underline{0}\right]$$

You did not select the correct vector field. The vector field given is the curl of the wanted vector field.

Marks for this submission: 0.00/1.00.

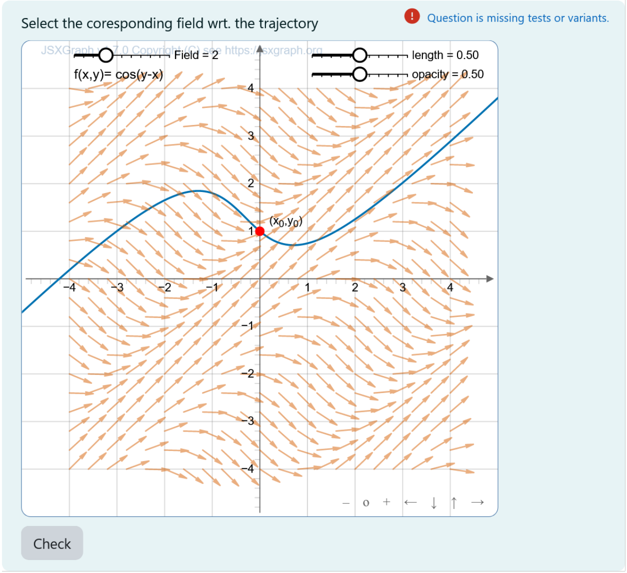
Feedback view

Slopefield and Trajectory

For a given ODE such as $y'(x) = f(x, y(x))$ with initial value $y(x_0) = y_0$ one can draw as the trajectory of the solution as the slope field given by $F(x, y) = \begin{pmatrix} 1 \\ f(x, y) \end{pmatrix}$.

The idea is to find the corresponding slope field for a given trajectory. JSXGraph provides the necessary tools, such as Runge-Kutta methods and the *slopefield* object.

ODE



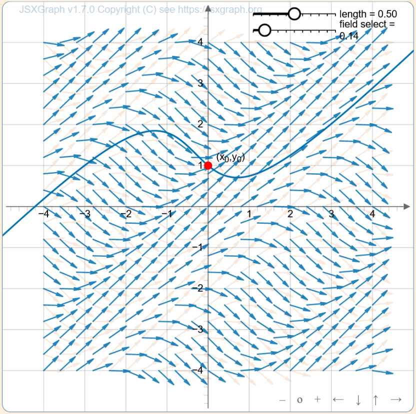
Students view

ODE

✕ Incorrect answer.

The selected field does not correspond with the trajectory shown.

You have selected the field $f(x,y) = \cos(y-x)$ (blue), the correct solution is $f(x,y) = -\sin(y-x)$ (red). Use the slider to see the differences of these fields. (0 - your answer, 1 - correct solution.)



Marks for this submission: 0.00/1.00.

Feedback view 1

ODE

✕ Incorrect answer.

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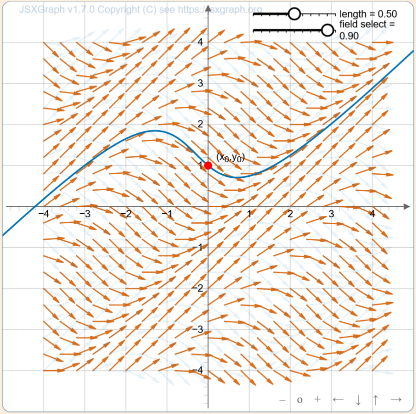


Marks for this submission: 0.00/1.00.

Feedback view 2

Constrained Optimisation and Lagrange Multiplier

The easiest constraint optimization problem in $\mathbb{R}^n$ is

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ s.t. & h(\mathbf{x}) = 0 \end{array}$$

The necessary optimality condition

> f, h are C^1 in a neighbourhood of $\mathbf{x}_0$ and $\nabla h(\mathbf{x}_0) \neq 0$ and $\mathbf{x}_0$ minimizes f subjected to $h(\mathbf{x}) = 0$.
> **Then** there exists a real number $\lambda \in \mathbb{R}$ with
>
> $$\nabla f(\mathbf{x}) + \lambda \nabla h(\mathbf{x}) = 0.$$

Necessary optimality condition seen geometrically

- $h(\mathbf{x}) = 0$ in JSXGraph 2D

A glider on a curve given implicitly by $h(\mathbf{x}) = 0$.

- $\nabla f(\mathbf{x}) + \lambda \nabla h(\mathbf{x}) = 0$

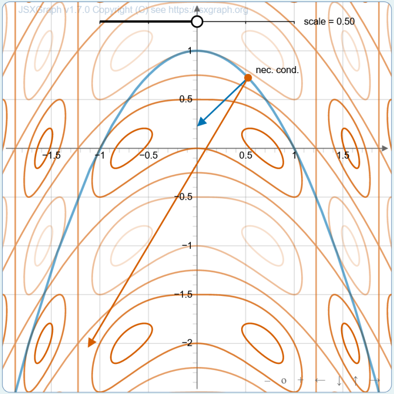
Two parallel vectors.

Lagrange Multiplier

Question is missing tests or variants.

Given are the contour lines of a function (red) and a constraint (blue).

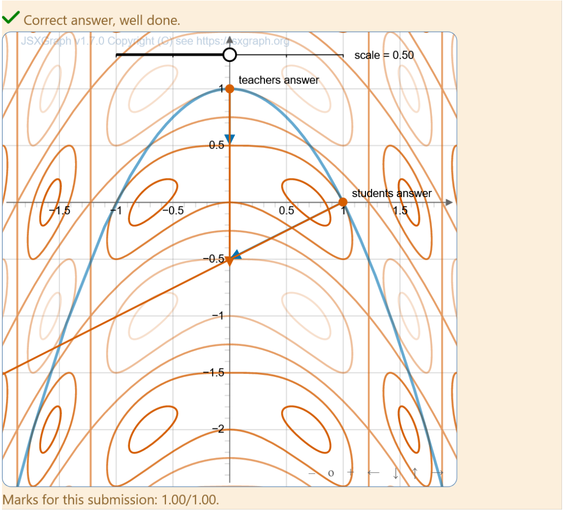
Find a point on the constraint fulfilling the necessary optimality condition



Check

Students view

Lagrange Multiplier



Feedback view

Transfer data: STACK to JSXGraph and back

» Numbers

```
tFinal = {#tFinalset#};
```

» strings, like function terms

```
funtxt = '{#fieldSelected#}';
fun = board.jc.snippet(frktxt, true, 'x');
```

The last line will create a JS function from the string stored in funtxt (see JSXGraph documentation JessieCode#snippet)

Sometimes it is needed to process the string using the *replace method*.

» Transfer of a list of strings can be done using the helper function *JXG.stack2jsxgraph*, e.g.

```
vecOfFields =
    JXG.stack2jsxgraph('{#listOfFields#}');
```

Strings can processed by e.g. *jc.snippet()*.

Why JSXGraph?

- streamline the applets
- fits my thinking coming from numerical math

demanding for developer

- documentation of the code for reuse
- minimizing the need to adapt code in JSXGraph to create modified questions
- initialize via questions variables