Visual e-Assessment with JSXGraph in Calculus

International meeting of the STACK community 2024, Amberg

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What is JSXGraph, who are we?

About us

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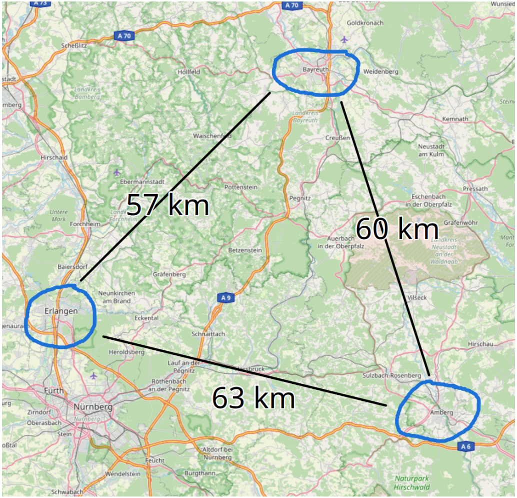
Lecturer for mathematics for engineers

Alfred Wassermann,
University of Bayreuth, Germany

JSXGraph lead developer

Where we are



57 km
60 km
63 km
Bayreuth
Erlangen
Amberg
Nürnberg
Fürth
Forchheim
Pegnitz
Weidenberg
Goldkronach
Hollfeld
Scheßlitz
Kemnath
Creußen
Ebermannstadt
Pottenstein
Betzenstein
Gräfenberg
Baiersdorf
Neunkirchen am Brand
Eckental
Schnaittach
Heroldsberg
Lauf an der Pegnitz
Röthenbach an der Pegnitz
Sulzbach-Rosenberg
Vilseck
Hirschau
Grafenwöhr
Eschenbach in der Oberpfalz
Auerbach in der Oberpfalz
Neustadt am Kulm
Pressath
Hirschaid
Waischenfeld
Altdorf bei Nürnberg
Feucht
Wendelstein
Burgthann
Schwabach
Zirndorf
Oberasbach
Stein
Naturpark Hirschwald
A 70
A 9
A 6

Why are we here?

> We represent *JSXGraph*, a library for “visualization of mathematics in web browsers”.
>
> Meanwhile, JSXGraph has become an important part of the STACK ecosystem and we are happy and proud that JSXGraph contributes to the success of STACK.
>
> Nevertheless, JSXGraph might be new to some of you. We will give an overview on JSXGraph and some ideas how to use it in e-assessment.

JSXGraph

> Interactive geometry, function plotting, charting, and data visualization in the web browser

- Open-source JavaScript library
- https://jsxgraph.org
- Developed at:
    - *Chair for Mathematics and Didactics* and
    - *Center for Mobile Learning with Digital Technology*, University of Bayreuth, Germany

Let’s start with examples

- Interactive geometry: Euler line
- Function plotting:
  - Riemann sums
  - Taylor sums
  - Elliptic curves
- Projective transformation: offside rule in soccer shared as *jsfiddle*

About JSXGraph

- JavaScript library:
  - constructions are “programmed” in JavaScript
  - JavaScript runs in every web browser
  - steep learning curve
- Open-source:
  - double license: MIT or LGPL
  - includes any commercial use
- Graphics quality:
  - the default graphics engine is SVG (vector graphic format)
- Performance:
  - small footprint: approx. 200kByte of bandwidth
  - it is fast!
- First JSXGraph release: 2008
- JSXGraph group: Carsten Miller, Andreas Walter, Alfred Wassermann

Ecosystem: support and contributions

Learn JSXGraph

- JSXGraph Book
- Wiki
- Examples database
- API reference
- Past workshops (slides and videos)

Getting and giving help

- Google Groups
- StackOverflow
- GitHub

List of features

- **Geometry:** points, lines, circles, conic sections, angles, sectors …
- **Plotting:** function graphs, parametric curves, polar curves, implicit curves
- **Interpolation:** Lagrange-polynomial, Bézier curves, various splines
- **Vector/slope fields**, differential equations
- Curve simplification, curve clipping
- Extensive library of (mostly numerical) **math functions**
- Some 3D
- Some symbolic math, e.g. derivatives

List of features, cont'd

- Seamless integration in web pages
  - Circles around circles
  - Project "everything is number"
  - A construction can contain arbitrary HTML elements, e.g. buttons and input fields
- Dynamic MathJax / KaTeX
- Images, videos

JSXGraph in moodle / ILIAS

JSXGraph filter for moodle and ILIAS

moodle

- Example
- Can be used in *all* text activities
- Syntax: <jsxgraph > ... </jsxgraph>
- Maintained by the JSXGraph group, available on the moodle plug-in page

ILIAS

- Status: a working proof-of-concept

Visual e-assessment

Wait a minute!

> It is nice that students can play around with constructions and function graphs, but can this feature be used for **assessment**, too?

» Consider this example: triangle area

> **Yes, it can:** $\rightarrow$ **STACK JSXGraph plug-in**

More examples of visual e-assessment

- Talk by *Carsten Miller* on Tuesday, 1:45pm session
- Sketch (cubic) function graphs (inspired by G. Kinnear, with M. Kallweit)
- Supported by various *Erasmus+ projects*: Compass, ITEMS, Expert, IDIAM

STACK JSXGraph plug-in

- Developed and maintained by *Matti Harjula*
- STACK example: triangle area, formative version
- Syntax: [[jsxgraph ...]] ... [[/ jsxgraph]]

Security

- since 2023, STACK has JSXGraph sandboxed in iframe
- Risk demo

Ideas for visual e-assessment in calculus

JSXGraph and STACK used in Calculus

- Some topics in engineering mathematics require the ability to visualise objects in 2D or 3D space.
- Multivariate analysis in the IDIAM project triggered the development of JSXGraph 3D.
  - Focus on integration domains (2D/3D) and extrema of functions in 2D

Ongoing ideas

- Work inspired by IDIAM
  - Application to implicit curves: constraint optimisation or Lagrange multiplier role
  - Slope fields and trajectories
- Examples available at the IDIAM Page
- JSXGraph 3D development and most examples have been funded by *ERASMUS+ "Interactive Digital Assessments in Mathematics"*.

Integration in 2D/3D

Given $G \subset \mathbb{R}^n$ $(n = 2, 3)$. The integral over G of a function $f : \mathbb{R}^n \to \mathbb{R}$ (integrable on G) is denoted as

$$\int_G f(\mathbf{x}) \, \mathrm{d}G.$$

How to compute this?

G may given by two functions

$$G = \{(x,y) \in \mathbb{R}^2 \,|\, a \leqslant x \leqslant b,\; y_2(x) \leqslant y \leqslant y_1(x)\}$$

then

$$\int\limits_G f(\mathbf{x}) \, \mathsf{d}G = \int\limits_a^b \int\limits_{y_2(x)}^{y_1(x)} f(x,y) \, \mathsf{d}y \mathsf{d}x.$$

> Recover the functions y_1, y_2 from a diagram is demanding for some students.

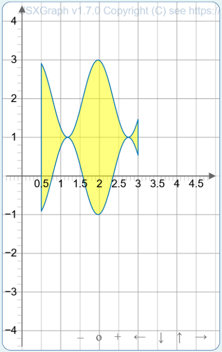
Given is a region of type

Question is missing tests or variants.

$$G = \{(x,y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ y_2(x) \leqslant y \leqslant y_1(x)\}$$

as shown in the diagram. Determine the interval $[a,b]$ and the expressions for the graphs of functions y_1 and y_2.

Give all numerical values as fractions instead of decimal numbers e.g. `1/2` instead of `0.5`.



$[a,b]$ =

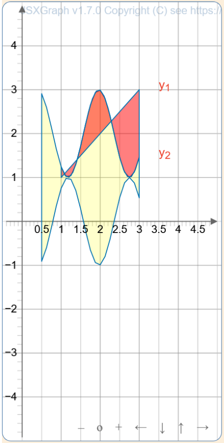
$y_1(x)$ =

$y_2(x)$ =

Check

Students view

This diagram shows the domain you have entered and the domain asked for. The given area is colored yellow, the red area results from your answer. The functions y_1 and y_2 have been labeled according to your input.



The value you gave for x_1 is not correct.

Nice, you found the correct value for x_2! Good job!

Check whether you did anything different here than for x_1 and try again.

Feedback view

Integration

The **Transformation Theorem** is widely used in integration.

> Given two sets $G \subset \mathbb{R}^n$ and $H \subset \mathbb{R}^n$ im $\mathbb{R}^n$ and a one-to-one mapping $T : H \to G$

$$T(\mathbf{u}) := \mathbf{x}(\mathbf{u}).$$

> T is continuously differentiable and $\det(J_T(u, v, w)) \neq 0$ on H. **Then**

$$\int_G f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_H f(\mathbf{x}(\mathbf{u})) \left|\det J_T(\mathbf{u})\right| \, \mathrm{d}\mathbf{u}.$$

Integration in 2D (Polar coordinates)

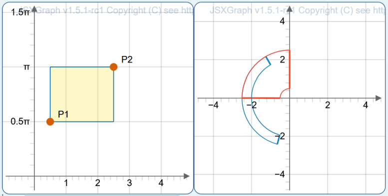
The introductory example in 2D integration are the Polar Coordinates:

$$T : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \text{ with } \begin{pmatrix} r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}$$

Given is a 2D area with polar geometry. It is defined by the intervals for each of the polar coordinates r and ϕ. Here r is the radial coordinate and ϕ is the angle starting at the x-axis oriented counterclockwise with $\phi \in [0, 2\pi]$. Tool zum Nachbessern der Frage | Es fehlen Tests oder Varianten.

Reconstruct the intervals that define the given area by matching the areas using the cartesian coordinate system.

Write the interval in the form $r \in$`[r1,r2]` and $\phi \in$`[phi1,phi2]`, e.g. `[1/2,2]` and `[1/2*pi,2*pi]`.



$r \in$

$\phi \in$

Students view

Integration in 3D

Spherical coordinates are very challenging for the students.

Question: Describe the set M given by

$$M := \{\mathbf{x} \in \mathbb{R}^3 \,:\, \|\mathbf{x}\|_2 \leqslant 1,\, x_1, x_2, x_3 < 0\}$$

in spherical coordinates.

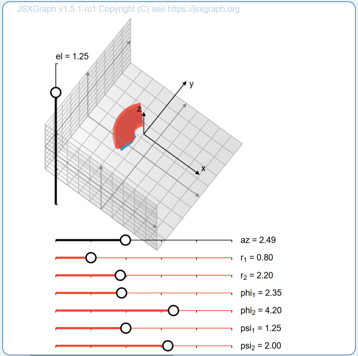
> Can only be solved by a few students.

Spherical Coordinates

Tool zum Nachbessern der Frage | Es fehlen Tests oder Varianten.

Given is a 3D volume with spherical geometry. It is defined by the intervals for each of the spherical coordinates r, ϕ and ψ. Here r is the radial coordinate and ϕ is the azimuthal angle starting at the x-axis oriented counterclockwise with $\phi \in [0, 2\pi]$. Lastly, ψ is the polar angle measured from the z-axis with $\psi \in [0, \pi]$.

Reconstruct the intervals that define the given volume.



$r_1 =$ 0.8

$r_2 =$ 2.2

$\phi_1 =$ 2.35

$\phi_2 =$ 4.2

$\psi_1 =$ 1.25

$\psi_2 =$

Students view

Spherical Coordinates

Your answer is partially correct.

The value you gave for r_1 is not correct.

Nice, you found the correct value for r_2! Good job!

Check whether you did anything different here than for r_1 and try again.

Correct answer, well done.

Nice, you found the correct value for ϕ_1! Good job!

Nice, you found the correct value for ϕ_2! Good job!

Perfect! You got both values of ϕ right!

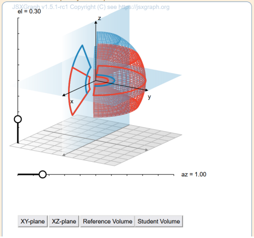
Your answer is partially correct.

The value you gave for ψ_1 is not correct.

Nice, you found the correct value for ψ_2! Good job!

Check whether you did anything different here than for ψ_1 and try again.

You can now compare your response to the reference volume. Your solution is displayed in orange. In addition, you can see the cross sections in the $x-y$-plane and $x-z$-plane. Note, that you can deactivate the visualizations using the button.



Feedback view

Curl

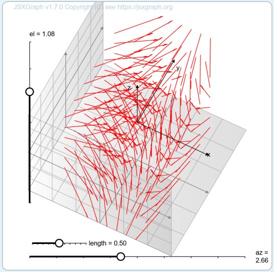
The curl of a vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ is just $\operatorname{curl} V = \nabla \times V$.

Question is missing tests or variants.

Given is the curl of a vector field $\hat{V} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$\hat{V}(x, y, z) := \begin{pmatrix} 1 - x \\ z \\ z \end{pmatrix}$ as shown in the diagram.

Select the vector field V, so that $\hat{V} = \nabla \times V$ is valid.



Select V.

- (Clear my choice)
- $[z, x, y]$
- $[y, -x, 0]$
- $\left[\frac{z^2}{2}, x \cdot z, y\right]$

$[y, -x, 0]$

Check

Curl

Incorrect answer.

The entries underlined in red below are those that are incorrect.

$$\left[\underline{y}, \underline{-x}, \underline{0}\right]$$

You did not select the correct vector field. The vector field given is the curl of the wanted vector field.

Marks for this submission: 0.00/1.00.

Feedback view

Slope field and Trajectory

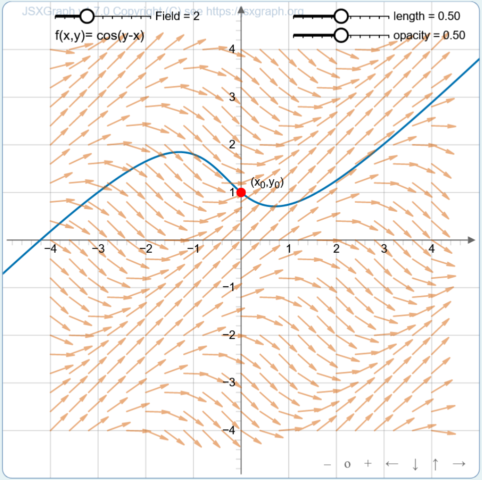
For a given ODE such as $y'(x) = f(x, y(x))$ with initial value $y(x_0) = y_0$ one can draw the trajectory of the solution as well as the slope field given by $F(x, y) = \begin{pmatrix} 1 \\ f(x, y) \end{pmatrix}$.

The idea is to find the corresponding slope field for a given trajectory. JSXGraph provides the necessary tools, such as Runge-Kutta methods and the *slopefield* object.

ODE

Select the coresponding field wrt. the trajectory

Question is missing tests or variants.



Check

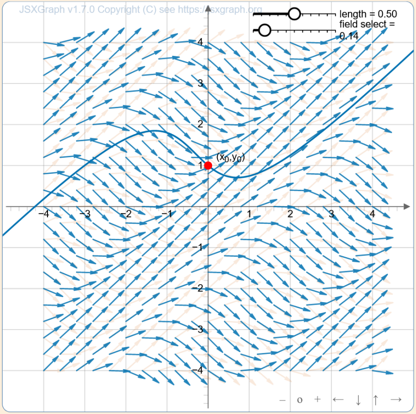
Students view

ODE

✗ Incorrect answer.

The selected field does not correspond with the trajectory shown.

You have selected the field $f(x, y) = \cos(y - x)$ (blue), the correct solution is $f(x, y) = -\sin(y - x)$ (red). Use the slider to see the differences of these fields. (0 - your answer, 1 - correct solution.)



Marks for this submission: 0.00/1.00.

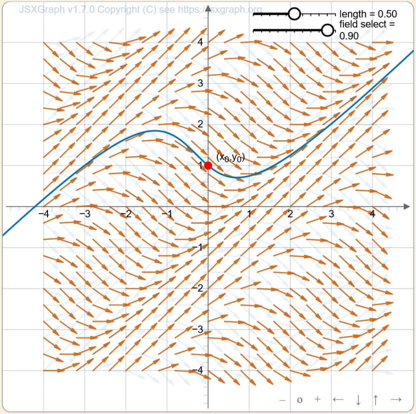
Feedback view 1

ODE

❌ Incorrect answer.

The selected field does not correspond with the trajectory shown.

You have selected the field $f(x,y) = \cos(y-x)$ (blue), the correct solution is $f(x,y) = -\sin(y-x)$ (red). Use the slider to see the differences of these fields. (0 - your answer, 1 - correct solution.)



Marks for this submission: 0.00/1.00.

Feedback view 2

Constrained Optimisation and Lagrange Multiplier

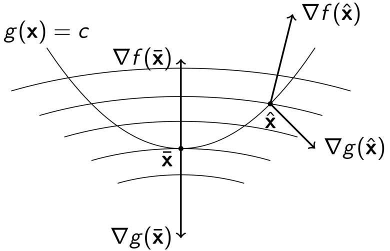
The easiest constraint optimization problem in $\mathbb{R}^n$ is

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ s.t. \quad & g(\mathbf{x}) = 0 \end{aligned}$$

Necessary first order optimality condition

> f, g are C^1 in a neighbourhood of $\mathbf{x}_0$ and $\nabla g(\mathbf{x}_0) \neq 0$
> and $\mathbf{x}_0$ minimizes f subjected to $g(\mathbf{x}) = 0$.
> **Then** there exists a real number $\lambda \in \mathbb{R}$ with
>
> $$\nabla f(\mathbf{x}) + \lambda \nabla g(\mathbf{x}) = 0.$$

Necessary optimality condition seen geometrically



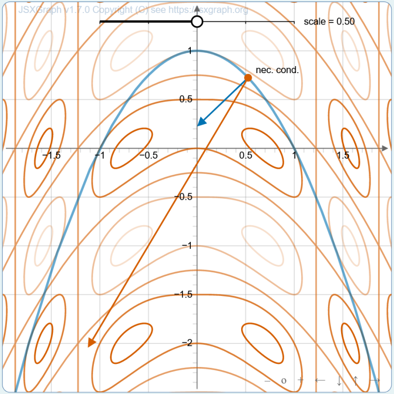
- $g(\mathbf{x}) = 0$ in JSXGraph 2D
  - A glider on a curve given implicitly by $g(\mathbf{x}) = 0$.
- $\nabla f(\mathbf{x}) + \lambda \nabla g(\mathbf{x}) = 0$
  - The two vectors are linearly dependent.

Lagrange Multiplier

Question is missing tests or variants.

Given are the contour lines of a function (red) and a constraint (blue).

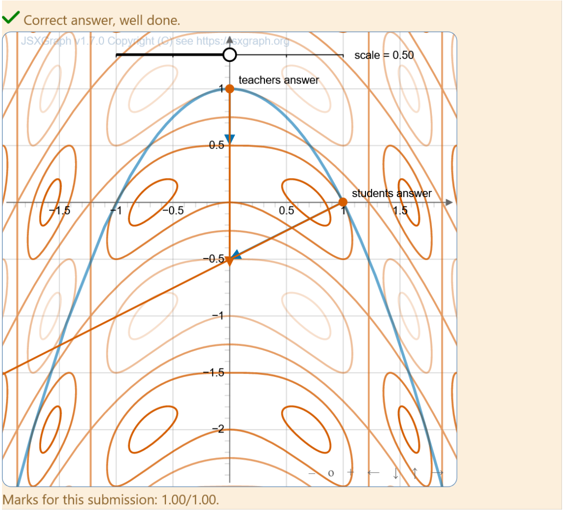
Find a point on the constraint fulfilling the necessary optimality condition



Check

Students view

Lagrange Multiplier



Feedback view

Transfer data: STACK to JSXGraph and back

» Numbers

```
tFinal = {#tFinalset#};
```

» Strings, like function terms:

```
funtxt = '{#fieldSelected#}';
fun = board.jc.snippet(funtxt, true, 'x');
```

The last line will create a JS function from the string stored in funtxt (see JSXGraph documentation JessieCode#snippet)

Sometimes it is necessary to process a string using the *replace method*.

» Transfer of a list of strings can be done using the helper function *JXG.stack2jsxgraph*, e.g.

```
vecOfFields =
    JXG.stack2jsxgraph('{#listOfFields#}');
```

» Strings can be converted to functions by e.g. *board.jc.snippet()*.

Final thoughts

Why JSXGraph?

» Streamline the applets
» Fits my thinking coming from numerical math

Recommendations for STACK authors

» Documentation of the code for reuse
» Minimizing the need to adapt code in JSXGraph to create modified questions
» Initialize via questions variables

Future work

STACK JXGraph plug-in

- Extend STACK bindings (feedback, please)

Projects

- Master the steep learning curve (in development):
  - JSXGraph author
  - Our take on mathematics on mobile devices: sketchometry, version 2beta will be released on March 20th.
  - Upcoming Typescript wrapper
- Graph theory with JSXGraph (source code)
- …

Closing

> We wish you a successful, fruitful conference.
>
> Please, get in contact with us!
>
> Remarks and questions?

Appendix

Communication STACK and JSXGraph

In our examples, we use

- basic features to transfer information from question variables to the JSXGraph block
- the STACK binding functionality to have access to student answers in PRTs.

Question variable $\rightarrow$ STACK

- STACK question variable: `functerm: sin(x)`
- `var graph = board.create('functiongraph', ['{#functerm#}']);`
- `var f = board.jc.snippet('{#functerm#}', true, 'x');`

JSXGraph $\rightarrow$ STACK

- `stack_jxg.bind_point(ans1Ref, A);`

Connection STACK - JSXGraph (binding)

» triangle area, formative version

```
Question text...
[[jsxgraph width="600px" height="600px"
    input-ref-ans1="ans1Ref"]]
// JSXGraph part ...
// STACK binding:
stack_jxg.define_group([A,B,C]);
stack_jxg.bind_list_of(ans1Ref, [A,B,C]);
stack_jxg.starts_moved(A);
[[/jsxgraph]]
<p style="display:none">
  [[input:ans1]][[validation:ans1]]
</p>
```

Alternative binding

» triangle area, summative version

```
// STACK binding:
var bindFunction = function(inpRef, watch, func) {
    let send = () => func();
    let receive = (val) => {};
    stack_jxg.custom_bind(inpRef, send, receive, watch);
};
bindFunction(ans1Ref, [A,B,C], areaFunction);
```

Communication STACK - JSXGraph

» There is a lot of freedom how to connect STACK and JSXGraph
» Documentation might deserve more examples